$$= F(n+1) - F(1) = \frac{e^{\frac{1}{(n+1)^2}} - e^{-1}}{2}$$

d'où $\forall n \in \mathbf{N}^*$, $\sum_{k=2}^{n+1} f(k) \leq \frac{e^{\frac{-1}{(n+1)^2}} - e^{-1}}{2} \leq \sum_{k=1}^{n} f(k)$

$$\Rightarrow \forall n \in \mathbf{N}^* \ \frac{e^{\frac{-1}{(n+1)^2}} - e^{-1}}{2} \leq \sum_{k=1}^{n} f(k) \Rightarrow \frac{e^{\frac{-1}{(n+1)^2}} - e^{-1}}{2n} \leq U_n$$

• Montrons que $\forall n \in \mathbf{N}^*$: $U_n \leq \frac{e^{\frac{-1}{n^2}} - e^{-1}}{2n}$

On a : $\forall n \in \mathbf{N}^*$, $\sum_{k=2}^{n+1} f(k) \leq \frac{e^{\frac{-1}{(n+1)^2}} - e^{-1}}{2} \Rightarrow$

$$\sum_{k=2}^{n} f(k) + f(n+1) \leq \frac{e^{\frac{-1}{(n+1)^2}} + e^{-1}}{2}$$

$$\Rightarrow \forall n \in \mathbf{N}^*, \sum_{k=2}^{n} f(k) \leq \frac{e^{\frac{-1}{n^2}} - e^{-1}}{2} \text{ car } f(n+1) > 0$$

$$\Rightarrow f(1) + \sum_{k=2}^{n} f(k) \leq \frac{e^{\frac{-1}{n^2}} - e^{-1}}{2} + f(1)$$

$$\Rightarrow \sum_{k=1}^{n} f(k) \leq \frac{e^{\frac{-1}{n^2}} - e^{-1}}{2} + e^{-1} \Rightarrow \frac{1}{n} \sum_{k=1}^{n} f(k) \leq \frac{e^{\frac{-1}{n^2}} + e^{-1}}{2n}$$

d'où $\forall n \in \mathbf{N}^*$, $U_n \leq \frac{e^{\frac{-1}{n^2}} + e^{-1}}{2n}$.

Conclusion : $\forall n \in \mathbf{N}^*$, $\frac{e^{\frac{-1}{(n+1)^2}} - e^{-1}}{2n} \leq U_n \leq \frac{e^{\frac{-1}{n^2}} + e^{-1}}{2n}$.